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Boundary operators and touching of loops in 2d gravity [1]

Masahiro Anazawa [2]

Department of Physics,
Graduate School of Science, Osaka University,
Toyonaka, Osaka, 560 Japan

Abstract

We investigate the correlators in unitary minimal conformal models coupled to two-dimensional gravity from the two-matrix model. We show that simple fusion rules for all of the scaling operators exist. We demonstrate the role played by the boundary operators and discuss its connection to how loops touch each other.

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[2] e-mail: anazawa@funpth.phys.sci.osaka-u.ac.jp JSPS research fellow

1 Introduction

The understanding of two-dimensional quantum gravity has experienced great progress through the study of the matrix models.[3] The one-matrix model has infinite number of critical points which are considered to represent the $(2m+1,2)$ minimal conformal models coupled to two-dimensional gravity. The two-matrix model has critical points which correspond to the $(m+1,m)$ unitary minimal conformal models [2, 3, 4]. In this paper we investigate the unitary minimal model $(m+1,m)$ coupled to two-dimensional gravity from the two-matrix model.

The emergence of the infinite number of scaling operators σ_j is one of the most important properties of the matrix models. Before coupled to gravity, the minimal model has finite number of primary fields. After gravitational dressing, however, infinite number of scaling operators emerge. This phenomenon can be understood as follows. In the Kac table we can divide the primary fields $\Phi_{r,s}$ into those which are inside the the minimal conformal grid $1 \le r \le q-1$, $1 \le s \le p-1$ and those outside, which correspond to the null states. Before dressed by gravity, the fields outside the minimal conformal grid decouple [5] from physical correlators. After gravitational dressing, they cease to decouple [6, 7] and become infinite number of scaling operators. The similar phenomenon has been shown in continuum framework. Through the examination of the BRST cohomology of the minimal model coupled to Liouville theory, infinite physical states were shown to exist [8]. These states have their counterparts in the matrix models as the scaling operators. Some of the scaling operators do not have their counterparts in the BRST cohomology, which we will discuss later.

In ordinary (p,q) minimal conformal model the primary fields satisfy certain fusion rules [5]; three-point function $\langle \Phi_{r_1,s_1}\Phi_{r_2,s_2}\Phi_{r_3,s_3}\rangle$ is non-vanishing only when

$$\begin{aligned} 1+|r_1-r_2| \le r_3 \le \mathrm{mim}\{r_1+r_2-1,p\}, \quad r_1+r_2+r_3 = \mathrm{odd} \\ 1+|s_1-s_2| \le s_3 \le \mathrm{mim}\{s_1+s_2-1,q\}, \quad s_1+s_2+s_3 = \mathrm{odd}\ . \end{aligned} \tag{1.1}$$

It is interesting to examine how the fusion rules change when the matter couples to gravity. The three-point functions involving lower dimensional scaling operators were examined from the point of view of the generalized KdV flow in [9] in the case of $(m+1,m)$ unitary matter. It was shown that the gravitational primaries σ_j $(j=1,\cdots,m-1)$ satisfy fusion rules of BPZ type; for $j_1+j_2+j_3 \le 2m-1$, $\langle \sigma_{j_1}\sigma_{j_2}\sigma_{j_3}\rangle$ is non-vanishing only when

$$1+|j_1-j_2| \le j_3 \le j_1+j_2-1\ . \tag{1.2}$$

The fusion rules were also examined in continuum framework [7]. As for the gravitational descendants, however, we think clear results have not been obtained. In

[3] See for example [1] for review.

this paper we would like to clarify the fusion rules for all of the scaling operators including the gravitational descendants in the case of unitary minimal model.

Macroscopic loop correlators, which are the amplitudes of the surfaces with boundaries (loops) of fixed lengths, are the fundamental amplitudes of the matrix models. It was shown [10, 11] that these correlators have more information than those of local operators and that the latter correlators can be extracted from the former correlators explicitly in the case of $c = 0, 1/2, 1$. They argued there that macroscopic loops could be replaced by a sum of local operators whose wave functions satisfy the Wheeler-DeWitt equations.

We generalize this idea to the loop correlators [4, 12, 13, 14] in the cases of the general unitary minimal models, and derive the fusion rules for all of the scaling operators. First we derive the explicit form of the expansion of loops in terms of the scaling operators, and then deduce the three-point correlators from the loop correlators which were calculated in [13, 14] from the two-matrix model. We show that the three-point correlators of all of the scaling operators satisfy certain simple fusion rules and these fusion rules are summarized in a compact form as the fusion rules for three-loop correlators [13].

In matrix models, there are infinite subset of the scaling operators $\widehat{\sigma}_j$ ($j = 0$ mod $m+1$) which do not have their counterparts in the BRST cohomology of the minimal model coupled to Liouville theory. In the case of one-matrix model, Martinec, Moore and Seiberg [15] argued that these operators are boundary operators which couple to the boundaries of two-dimensional surface. They proved that one of them is in fact a boundary operator which measures the total loop length. But little has been discussed on the role played by the rest of these operators. We examine the geometrical meaning of these operators and its connection to the touching of loops in the case of general unitary models. We show that the boundary operators have the role to connect several parts of the loops together. We also discuss the relation of the boundary operators to the Schwinger-Dyson equations proposed in [17].

2 Expansion of loops in local operators

We consider the $(m+1, m)$ unitary minimal model coupled to two-dimensional gravity from the two-matrix model with symmetric potential. The partition function Z is defined by

$$\mathrm{e}^{Z} = \int \mathrm{d}\hat{A}\mathrm{d}\hat{B}\ \mathrm{e}^{-\frac{N}{\Lambda}\mathrm{tr}\left(U(\hat{A})+U(\hat{B})-\hat{A}\hat{B}\right)} \ , \tag{2.1}$$

where $\hat{A}$ and $\hat{B}$ are hermitian matrices, and U is a certain polynomial. In this article, we limit our discussion to the two-matrix model with symmetric potential and to the critical points which correspond to the unitary minimal models. In the case

of asymmetric potential, some of the boundaries (loops) of two-dimensional surface would have fractal dimensions different from the usual dimension of length.

In [10], in the case of the one-matrix model, it was shown that the loop operator can be expanded in terms of local operators, that is, the loop can be replaced with the infinite combination of local operators, except some special cases. It was argued that this is the case for the general $(m+1,m)$ unitary models. Whether this replacement can be done safely or not is connected with whether the corresponding classical solution exits or not in the limit of small length of the corresponding loop. This claim is quite natural because all of the scaling operators are expressed in term of one matrix $\hat{A}$ as $\sigma_j = \mathrm{Tr}(1-\hat{A})^{j+1/2} = \sum_n a_n(j) n^{-1}\,\mathrm{Tr}\,\hat{A}^n$ in the one-matrix model.

In the two-matrix model, it appears that this idea is not the case since the direct connection of the scaling operators to the operators $\mathrm{Tr}\,\hat{A}^n$ or $\mathrm{Tr}\,\hat{B}^{n'}$ is not clear. But this expansion is considered to be possible by the following reason. When one of the loops on two-dimensional surface shrunk to a microscopic loop, the loop represents local deformation of the surface. The microscopic loop can be replaced by the insertions of local operators. The loop correlators except one-loop case are continuous when the length of one of the loops approaches zero, so that we expect that a macroscopic loop can also be replaced by a sum of local operators.

In this section we derive explicitly the expansion of loops in local operators in the case of the unitary minimal models. Using this relation, we can deduce the amplitudes of local operators and those involving both loops and local operators from the loop correlators.

Let us recall that the two-loop correlators in the $(m+1,m)$ unitary minimal model coupled to two-dimensional gravity are [4, 13]

$$\begin{aligned}&\left\langle w^+(\ell_1) w^\pm(\ell_2)\right\rangle \\ &\quad = \frac{1}{m}\frac{M}{2}\frac{\ell_1\ell_2}{\ell_1+(\pm)^m\ell_2}\sum_{k=1}^{m-1}(\pm)^{k-1}\widetilde{K}_{\frac{k}{m}}(M\ell_1)\,\widetilde{K}_{1-\frac{k}{m}}(M\ell_2) \quad . \end{aligned} \tag{2.2}$$

for $\ell_1 \neq \ell_2$. Here $w^+(\ell)$ and $w^-(\ell)$ represent loop operators which create loops made by the matrices $\hat{A}$ and $\hat{B}$ respectively, and we introduced a notation

$$\widetilde{K}_p(M\ell) = \frac{\sin\pi|p|}{\pi/2}K_p(M\ell)\ . \tag{2.3}$$

Here the parameter M is defined by

$$\left(\frac{M}{2}\right)^2 = \frac{\mu}{m+1}, \quad \Lambda - \Lambda_* = -a^2\mu, \tag{2.4}$$

where Λ_* represents the critical value of the bare cosmological constant, and μ is the renormalized cosmological constant.

The definition of a set of the scaling operators has arbitrariness which comes from the contact terms [10]. As a set of local operators, we take the scaling operators $\widehat{\sigma}_j$ whose wave functions $\Psi_j(\ell)$ satisfy the (minisuperspace) Wheeler-DeWitt equations

$$\left[-\left(\ell \frac{\partial}{\partial \ell}\right)^2 + 4\mu\ell^2 + \left(\frac{j}{m}\right)^2 \right] \Psi_j(\ell) = 0 \ . \tag{2.5}$$

It was shown [10] that these scaling operators correspond to the dressed primary fields of the conformal field theory in the case of one-matrix model. In terms of these scaling operators, we can obtain simple fusion rules for three-point correlators, which we will show in the next section.

We normalize the wave function of $\widehat{\sigma}_j$ as

$$\left\langle \widehat{\sigma}_j \; w^+(\ell) \right\rangle = \frac{j}{m} \left(\frac{M}{2}\right)^{\frac{j}{m}} \; \widetilde{K}_{\frac{j}{m}}(M\ell) \ \ , \ \ j \geq 1 \ \neq 0 \ (\mathrm{mod}\ m) \ \ . \tag{2.6}$$

Note that the normalization factor $\sin \frac{j}{m}\pi$ in $\widetilde{K}_j(M\ell)$ in eq. (2.6) is consistent because there are no scaling operators $\widehat{\sigma}_j$ for $j = 0$ (mod m) in the matrix model. We can express the right hand side of eq. (2.2) as an infinite sum in terms of $\widetilde{K}_j(M\ell_2)$ for $\ell_1 < \ell_2$ (see appendix A):

$$\begin{aligned} &\left\langle w^+(\ell_1) w^\pm(\ell_2) \right\rangle \\ &\quad = \frac{1}{m} \sum_{k=1}^{m-1} \sum_{n=-\infty}^{\infty} (\pm)^{k-1} \left| \frac{k}{m} + 2n \right| \; I_{|\frac{k}{m}+2n|}(M\ell_1) \; \widetilde{K}_{\frac{k}{m}+2n}(M\ell_2) \ . \end{aligned} \tag{2.7}$$

Comparing eq. (2.6) with eq. (2.7), we expect the following expansions of the loop operators in term of the local operators:

$$\text{“} \ w^\pm(\ell) = \frac{1}{m} \sum_{k=1}^{m-1} \sum_{n=-\infty}^{\infty} (\pm)^{k-1} \left(\frac{M}{2}\right)^{-|\frac{k}{m}+2n|} \quad I_{|\frac{k}{m}+2n|}(M\ell) \; \widehat{\sigma}_{|k+2mn|} \ \text{”} \ . \tag{2.8}$$

These expansions are the generalizations of those in the case of the one-matrix model [10] to the cases of general unitary minimal models coupled to two-dimensional gravity.

Since the loop correlators are symmetric under the interchange of two kinds of loops, that is, $\langle w^+(\ell_1) \; w^+(\ell_2) \rangle = \langle w^-(\ell_1) \; w^-(\ell_2) \rangle$, the wave functions of the scaling operators with respect to loop $w^-(\ell)$ are read as

$$\left\langle \widehat{\sigma}_j \; w^-(\ell) \right\rangle = (-1)^{j-1} \left\langle \widehat{\sigma}_j \; w^+(\ell) \right\rangle \ \ . \tag{2.9}$$

3 Fusion rules for scaling operators

Using the expansion of loops eq. (2.8), we can obtain the correlators of the scaling operators from loop correlators.[4] In this section, we show that there are rather simple fusion rules for all of the scaling operators. The fusion rules involving the gravitational descendants ($\sigma_j,\ j \geq m+2$) have not been clear from the point of view of generalized KdV flow.

3.1 One- and two-point functions

Let us examine one- and two-point functions first. Since the one-loop amplitude diverges when the loop length approaches to zero, this amplitude include the contribution which is not represented by the local operators. Extracting the parts proportional to I_ν ($\nu > 0$), which parts can be considered as the contributions from local operators, from the one-loop amplitude

$$\begin{aligned}\left\langle w^{\pm}(\ell)\right\rangle &= \left(1+\frac{1}{m}\right)\ell^{-1}\left(\frac{M}{2}\right)\widetilde{K}_{1+\frac{1}{m}}(M\ell)\\ &= \left(\frac{M}{2}\right)^{2+\frac{1}{m}}\left(I_{2+\frac{1}{m}}(M\ell) - I_{-2-\frac{1}{m}}(M\ell) - I_{\frac{1}{m}}(M\ell) + I_{-\frac{1}{m}}(M\ell)\right)\ ,\end{aligned} \tag{3.1}$$

we can obtain the one-point functions of the scaling operators

$$\langle\widehat{\sigma}_1\rangle = -m\left(\frac{M}{2}\right)^{2+\frac{2}{m}}\ ,\ \ \langle\widehat{\sigma}_{1+2m}\rangle = m\left(\frac{M}{2}\right)^{4+\frac{4}{m}}\ , \tag{3.2}$$

$$\langle\widehat{\sigma}_j\rangle = 0\ \ ,\ \ j \neq 1, 1+2m\ . \tag{3.3}$$

Let us turn to the two-point functions. Substituting eq. (2.8) into eq. (2.6), we obtain the two-point functions

$$\langle\widehat{\sigma}_i\widehat{\sigma}_j\rangle = \delta_{ij}\ j\ \left(\frac{M}{2}\right)^{2j/m}\ \ ,\ \ i,j \neq 0\ (\mathrm{mod}\ m)\ \ . \tag{3.4}$$

Note that we obtain diagonal two-point functions.

3.2 Three-point functions

The three-loop correlator from two-matrix model is obtained in [13] as

$$\left\langle w^{+}(\ell_1)w^{+}(\ell_2)w^{+}(\ell_3)\right\rangle = -\frac{1}{m(m+1)}\left(\frac{M}{2}\right)^{1-\frac{1}{m}}\ell_1\ell_2\ell_3$$

[4]The multi-loop correlators were examined in [14] from two-matrix model. These correlators were also examined in [16] from the viewpoint of random surfaces immersed in Dynkin diagrams.

$$\times \sum_{\substack{(k_1-1,k_2-1,k_3-1)\\ \in\mathcal{D}_3^{(m)}}} \widetilde{K}_{1-\frac{k_1}{m}}(M\ell_1)\,\widetilde{K}_{1-\frac{k_2}{m}}(M\ell_2)\,\widetilde{K}_{1-\frac{k_3}{m}}(M\ell_3)\ , \tag{3.5}$$

Here we have denoted by $\mathcal{D}_3^{(m)}$

$$\begin{aligned}\mathcal{D}_3^{(m)} \quad = \quad & \Bigg\{(a_1,a_2,a_3)\ \Big|\ \sum_{i(\neq j)}^{3} a_i - a_j \geq 0 \text{ for } j=1\sim 3\ ,\\ & \sum_{i=1}^{3} a_i = \text{even} \leq 2(m-2)\ , a_i = 0,1,2,\cdots \Bigg\}\ .\end{aligned} \tag{3.6}$$

It was shown [13] that the selection rules in eqs. (3.5) and (3.6) correspond to the fusion rules [9, 7] for the dressed primaries ($\phi_{ii},\ i=1,\cdots,m-1$) by studying the small length behavior of the three-loop correlator. Using the expansion of loop eq. (2.8), we can show that the selection rules in eq. (3.5) represent the fusion rules for all of the scaling operators in a compact form. Let us show this in the following.

Using the formula,

$$zK_{1-|p|}(z) = \pi \sum_{n=-\infty}^{\infty} \frac{|p+2n|}{\sin\pi|p+2n|} I_{|p+2n|}(z)\ \ , \tag{3.7}$$

we first expand the three-loop correlator eq. (3.5) as

$$\begin{aligned}\Big\langle w^+(\ell_1)w^+(\ell_2)w^+(\ell_3)\Big\rangle = \frac{-1}{m(m+1)}\left(\frac{M}{2}\right)^{-2-\frac{1}{m}} \sum_{\mathcal{D}_3^{(m)}}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\sum_{n_3-\infty}^{\infty}\\ \left(\tfrac{k_1}{m}+2n_1\right)\left(\tfrac{k_2}{m}+2n_2\right)\left(\tfrac{k_3}{m}+2n_3\right) I_{|\frac{k_1}{m}+2n_1|}(M\ell_1) I_{|\frac{k_2}{m}+2n_2|}(M\ell_2) I_{|\frac{k_3}{m}+2n_3|}(M\ell_3)\ .\end{aligned} \tag{3.8}$$

Comparing eq. (3.8) with eq. (2.8), we can extract the three-point functions

$$\begin{aligned}&\Big\langle \widehat{\sigma}_{|k_1+2mn_1|}\widehat{\sigma}_{|k_2+2mn_2|}\widehat{\sigma}_{|k_3+2mn_3|}\Big\rangle\\ &\quad = C_{k_1k_2k_3}\frac{-1}{m(m+1)}\prod_{i=1}^{3}(k_i+2mn_i)\left(\frac{M}{2}\right)^{-2-\frac{1}{m}+\sum_{i=1}^{3}\frac{1}{m}|k_i+2mn_i|}\ ,\end{aligned} \tag{3.9}$$

where

$$C_{k_1k_2k_3} = \begin{cases} 1\ , & (k_1-1,k_2-1,k_3-1)\in\mathcal{D}_3^{(m)} \\ 0\ , & \text{otherwise}\end{cases}\ . \tag{3.10}$$

For $n_i = 0$, eq. (3.9) is nothing but the correlator of the gravitational primaries. For the gravitational primaries, eq. (3.4) and eq. (3.9) agree with the correlators obtained in [9] from the generalized KdV flow up to a factor -2. Note that we obtain, here, the correlators of the gravitational descendants as well. In [7], the fusion

rules for the gravitational primaries were examined in continuum framework. We have found here the fusion rules for the gravitational descendants as well as for the gravitational primaries. These fusion rules are similar to those for the gravitational primaries due to the factor $C_{k_1k_2k_3}$ in eq. (3.9).

Introducing the equivalence classes $[\widehat{\sigma}_k]$ by the equivalence relation

$$\widehat{\sigma}_k \sim \widehat{\sigma}_{|k+2mn|} \quad , \quad n \in \mathbb{Z} \quad , \tag{3.11}$$

we can consider the fusion rules in eq. (3.9) as fusion rules among $[\widehat{\sigma}_k]$ ($k = 1, \cdots, m-1$). Note here that the class $[\widehat{\sigma}_k]$ does not correspond to the set which consist of the gravitational primary $\mathcal{O}_k$ and its gravitational descendants $\sigma_l\left(\mathcal{O}_k\right)$, $l = 1, 2, \cdots$ in [9] introduced from the viewpoint of KdV flow.

3.3 Further on the fusion rules

In this subsection, let us examine the fusion rules in eq. (3.9) further and summarize the relation of the scaling operators to the primary fields in the corresponding conformal field theory.

In the (p, q) minimal conformal model, the primary field Φ_{rs} has the conformal dimension

$$\Delta_{r,s} = \frac{(pr - qs)^2 - (p-q)^2}{4pq} \quad , \tag{3.12}$$

where r and s are positive integers. Since we have $\Delta_{r,s} = \Delta_{r+q,s+p} = \Delta_{q-r,p-s}$, the corresponding primary fields can be regarded as the same one. The integers r and s can thus be restricted in the range

$$1 \leq r \leq q-1, \quad 1 \leq s \neq 0 \; (\mathrm{mod}\; p), \quad pr < qs \tag{3.13}$$

(see fig. 1). In fig. 1, the primary fields in the region ⦅2⦆ or ⦅2⦆′ are the secondary fields of those in the region ⦅1⦆. In general, the fields in the region ⦅n+1⦆ or ⦅n+1⦆′ are the secondaries of the fields in ⦅n⦆ or ⦅n⦆′. Since the secondary fields correspond to null vectors, those fields decouple. One can thus construct consistent conformal field theory which include only the primary fields in the region ⦅1⦆ (i.e. inside the minimal table), that is, the (p, q) minimal model [5]. Coupled to Liouville theory, however, the fields outside the the minimal table fail to decouple [6] and infinite physical states emerge accordingly. These states are considered to correspond to the primaries outside the minimal table. This correspondence is implied by the BRST cohomology [8] of the coupled system.

In the minimal model coupled to Liouville theory, the gravitational dimension $\Delta^G_{r,s}$ of the dressed operator for $\Phi_{r,s}$ is given by the relations

$$\Delta^G_{r,s} = 1 - \frac{\alpha_{r,s}}{\gamma} \; , \quad \frac{\alpha_{r,s}}{\gamma} = \frac{p + q - |pr - qs|}{2q} \; , \tag{3.14}$$

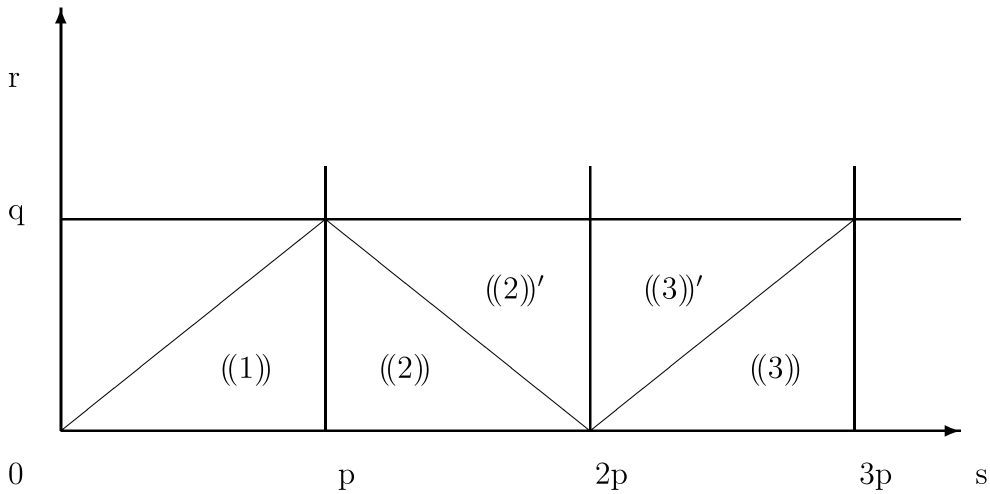

Figure 1: the range of (r, s)

where r and s take the values in the range eq. (3.13). On the other hand, in the matrix model, the corresponding relation for the scaling operator $\widehat{\sigma}_j$ is given by

$$\frac{\alpha_j}{\gamma} = \frac{p+q-j}{2q} \quad . \tag{3.15}$$

From eq. (3.14) and eq. (3.15), we should take as

$$j = |pr - qs| \ , \ \ j = 1, 2, \cdots \neq 0 \ (\mathrm{mod}\ q) \ , \tag{3.16}$$

for $\widehat{\sigma}_j$.

Consider now the relation of $\widehat{\sigma}_{|k+2mn|}$ to the primary field $\Phi_{r,s}$ of the unitary $(m+1, m)$ minimal model. Let us first compare the two sets

$$S_k = \left\{ |k + 2nm| \ \mid \ n \in \mathbb{Z} \right\} \ , \tag{3.17}$$

and

$$\begin{aligned} \left\{ \ |pr - qs| \ \right\} \ &= \ \left\{ \ |(m+1)r - ms| \ \right\} \\ &= \ \left\{ \ r' + (s - r - 1)m \ \right\} \ , \end{aligned} \tag{3.18}$$

where r and s are positive integers in the range

$$1 \leq r \leq m-1, \quad 1 \leq s, \quad r+1 \leq s \tag{3.19}$$

and $r' \equiv m - r$. Note that we include $s = 0$ (mod $m+1$) here. Decomposing the set S_k into two sets as

$$S_k = S_k^+ \oplus S_k^- \ , \tag{3.20}$$

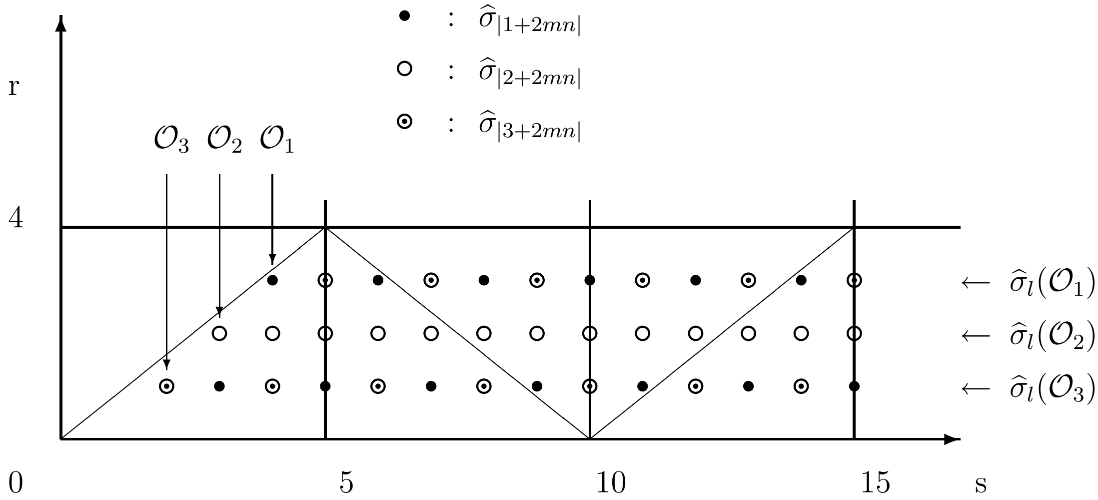

Figure 2: the scaling operators $\widehat{\sigma}_{|k+2mn|}$ in the $(5,4)$ minimal model coupled to 2d gravity

where

$$\begin{aligned} S_k^+ &= \Big\{ k + 2nm \ \mid \ n = 0,1,2,\cdots \Big\} \ , \\ S_k^- &= \Big\{ (m-k) + (2n'+1)m \ \mid \ n' = 0,1,2,\cdots \Big\} \ , \end{aligned} \tag{3.21}$$

and comparing eq. (3.21) and eq. (3.18), we can express the sets S_k^+ and S_k^- in terms of $|(m+1)r - ms|$ as

$$\begin{aligned} S_k^+ &= \Big\{ \ |(m+1)r - ms| \ \mid \ r' = k, \ s - r = 2n+1, \ n = 0,1,\cdots \Big\} , \\ S_k^- &= \Big\{ \ |(m+1)r - ms| \ \mid \ r = k, \ s - r = 2n'+2, \ n' = 0,1,\cdots \Big\} . \end{aligned} \tag{3.22}$$

From eq. (3.22), the following correspondence is obtained:

$$\begin{aligned} \widehat{\sigma}_{|k+2mn|} \ (n \geq 0) &\leftrightarrow \Phi_{m-k,\, r+2n+1} \ (n \geq 0) \\ \widehat{\sigma}_{|k+2m(-1-n')|} \ (n' \geq 0) &\leftrightarrow \Phi_{k,\, r+2n'+2} \ (n' \geq 0) \ , \end{aligned} \tag{3.23}$$

where $s \neq 0 \pmod{m+1}$. As for the scaling operators corresponding to $s = 0 \pmod{m+1}$, we will discuss these in the next section.

As an example, we depicted the scaling operators on the r-s plane for the case of $m = 4$ in fig. 2. In this figure we showed the equivalence classes $[\widehat{\sigma}_k]$ explicitly.

4 Boundary operators

4.1 Boundary operators and touching of loops

The scaling operators $\widehat{\sigma}_j$ ($j = 0$ mod $m+1$), which correspond to $s = 0$ (mod $m+1$) on the r-s plane do not have their counterparts in the BRST cohomology of the minimal model coupled to Liouville theory. In [15] it was proposed that the scaling operators which do not occur in the BRST cohomology of Liouville theory are boundary operators and one of them, which is $\widehat{\sigma}_3 = \widehat{\sigma}_1(\mathcal{O}_1)$ in the case of pure gravity, was in fact proven to be a boundary operator for the one-matrix model and the Ising model case. We would like to examine the role played by the operators $\widehat{\sigma}_{n(m+1)}$, $n = 1, 2, \cdots \neq 0$ (mod m) as well as $\widehat{\sigma}_{m+1}$ for general unitary minimal models.

Let us denote these operators by

$$\widehat{\mathcal{B}}_n = \widehat{\sigma}_{n(m+1)}, \quad n = 1, 2, \cdots \neq 0 \text{ (mod } m) \ . \tag{4.1}$$

In the matrix models the loop amplitudes contain the contribution from the surfaces with loops touching each other. In two-loop case, let us consider the surfaces in which the two loops touch each other on n points. When we shrink one of the loops to a microscopic loop, the other loop splits into n loops, which are stuck each other through the microscopic loop (see figs.3, 4 and 5). Since the microscopic loop represents a sum of the scaling operators, the wave functions of some scaling operators contain the contribution from the surfaces with split loop.

We now show that the boundary operators indeed represent these surfaces. From eqs. (2.6) and (3.1), the wave function of $\widehat{\mathcal{B}}_n$ and the one-loop amplitude are

$$\left\langle \widehat{\mathcal{B}}_n w^+(\ell) \right\rangle = n(1+\frac{1}{m}) \left(\frac{M}{2}\right)^{n(1+\frac{1}{m})} \widetilde{K}_{n(1+\frac{1}{m})}(M\ell) \ , \tag{4.2}$$

$$\left\langle w^+(\ell) \right\rangle = (1+\frac{1}{m}) \ \ell^{-1} \left(\frac{M}{2}\right)^{1+\frac{1}{m}} \widetilde{K}_{1+\frac{1}{m}}(M\ell) \ . \tag{4.3}$$

We denote the Laplace transformation of any function $f(\ell_1, \ell_2, \cdots)$ of loop lengths by

$$\mathcal{L}\left[f(\ell_1, \ell_2, \cdots)\right] = \int_0^\infty d\ell_1 \int_0^\infty d\ell_2 \cdots e^{-(\ell_1\zeta_1 + \ell_2\zeta_2 + \cdots)} f(\ell_1, \ell_2, \cdots) \ . \tag{4.4}$$

In the space of Laplace transformed coordinate ζ, we have

$$\mathcal{L}\left[\ell^{-1}\langle \widehat{\mathcal{B}}_n w^+(\ell)\rangle\right] = -\left(\frac{M}{2}\right)^{n(1+\frac{1}{m})} 2\cosh n(m+1)\theta \ , \tag{4.5}$$

$$\mathcal{L}\left[\langle w^+(\ell)\rangle\right] = -\left(\frac{M}{2}\right)^{1+\frac{1}{m}} 2\cosh(m+1)\theta \ , \tag{4.6}$$

where we have used the relation

$$\mathcal{L}\left[-\ell^{-1}|\nu|\widetilde{K}_\nu(M\ell)\right] = 2\cosh m\nu\theta \ , \tag{4.7}$$

and ζ is parametrized as $\zeta = M\cosh m\theta$. Note here that $w^+(\ell)$ represents a loop with a marked point and $\ell^{-1}w^+(\ell)$ represents a loop without a marked point. Since $\cosh n(m+1)\theta$ can be expressed as a polynomial of $\cosh(m+1)\theta$,

$$\begin{aligned} 2\cosh n(m+1)\theta &= 2\ T_n\Big(\cosh(m+1)\theta\Big) \\ &= \sum_{r=0}^{[n/2]} c_r^{(n)}\Big[2\cosh(m+1)\theta\Big]^{n-2r}, \quad c_r^{(n)} = \frac{(-1)^r n}{n-r}\binom{n-r}{r} \end{aligned} \tag{4.8}$$

where T_n is the Chebeyshev polynomial, we obtain the following relation:

$$\mathcal{L}\left[-\ell^{-1}\langle\widehat{\mathcal{B}}_n w^+(\ell)\rangle\right] = \sum_{r=0}^{[(n-1)/2]} c_r^{(n)}\left(\frac{M}{2}\right)^{2r(1+\frac{1}{m})}\Big\{\mathcal{L}\left[-\langle w^+(\ell)\rangle\right]\Big\}^{n-2r} \ . \tag{4.9}$$

In the space of loop length, the above relation means that the wave function of $\widehat{\mathcal{B}}_n$ is equivalent to a sum of the convolutions of disk amplitudes:

$$\Big\langle\widehat{\mathcal{B}}_n w^+(\ell)\Big\rangle = -\ell\sum_{r=0}^{[(n-1)/2]} c_r^{(n)}\left(\frac{M}{2}\right)^{2r(1+\frac{1}{m})}(-1)^{n-2r}\left[\odot\mathcal{A}_1^+\right]^{n-2r}(\ell) \ . \tag{4.10}$$

Here we introduced a notation $\mathcal{A}_1^+ \equiv \langle w^+(\ell)\rangle$, and $\left[\odot\mathcal{A}_1^+\right]^s(\ell)$ denotes the convolution of s $\mathcal{A}_1^+(\ell)$'s, for example

$$\left[\odot\mathcal{A}_1^+\right]^2(\ell) = \int_0^\infty\int_0^\infty \mathrm{d}\ell_1\mathrm{d}\ell_2\ \delta(\ell_1+\ell_2-\ell)\mathcal{A}_1^+(\ell_1)\mathcal{A}_1^+(\ell_2) \ . \tag{4.11}$$

From eq. (4.10) we can conclude that the operator $\widehat{\mathcal{B}}_n$ couple to the point to which s ($\leq n$) parts of the loop are stuck each other in the case of one-loop amplitudes. When there are more than one loop, we infer that the operator couples to the point to which s parts of several loops are stuck each other; the operator will not recognize that it is touching different loops this time.

Using the following relation

$$\Big[2\cosh x\Big]^n = \sum_{r=0}^{[(n-1)/2]}\binom{n}{r}2\cosh(n-2r)x, \ \ (\text{up to constant}) \ , \tag{4.12}$$

we also obtain

$$\ell\left[\odot\mathcal{A}_1^+\right]^n(\ell) = (-1)^{n+1}\sum_{r=0}^{[(n-1)/2]}\binom{n}{r}\left(\frac{M}{2}\right)^{2r(1+\frac{1}{m})}\Big\langle\widehat{\mathcal{B}}_{n-2r}w^+(\ell)\Big\rangle \ . \tag{4.13}$$

Here we drop the constant term in eq. (4.12) when we carry out the inverse Laplace transformation. From eq. (4.13), we see that the boundary operator coupled to the point on which n parts of loops are touching each other is given by

$$\mathcal{B}_n = (-1)^{n+1} \sum_{r=0}^{[(n-1)/2]} \binom{n}{r} \left(\frac{M}{2}\right)^{2r(1+\frac{1}{m})} \widehat{\mathcal{B}}_{n-2r} \ . \tag{4.14}$$

Now let us consider the boundary operators when there are two loops on two-dimensional surface. As for $\mathcal{B}_1$, we expect that $\langle w^+(\ell_1)w^+(\ell_2)\mathcal{B}_1\rangle$ should be proportional to $(\ell_1+\ell_2)\langle w^+(\ell_1)w^+(\ell_2)\rangle$. Let us confirm this in the following. From the three loop correlator (3.5), the expansion of loop operator (2.8) and the wave function of $\widehat{\sigma}_{|k+2mn|}$ (2.6), we obtain the following correlator with two loops and a local operator:

$$\begin{aligned}\left\langle w^+(\ell_1)w^\pm(\ell_2)\widehat{\sigma}_{|k_3+2mn_3|}\right\rangle &= \frac{-1}{m+1}\sum_{k_1,k_2} C_{k_1k_2k_3}(\pm)^{k_2-1}\left(\frac{M}{2}\right)^{-\frac{1}{m}+|\frac{k_3}{m}+2n_3|} \\ &\times \ell_1\ell_2(\frac{k_3}{m}+2n_3)\widetilde{K}_{1-\frac{k_1}{m}}(M\ell_1)\widetilde{K}_{1-\frac{k_2}{m}}(M\ell_2) \ .\end{aligned} \tag{4.15}$$

Consider the amplitude for $\mathcal{B}_1 = \widehat{\mathcal{B}}_1 = \widehat{\sigma}_{m+1} = \widehat{\sigma}_{|\frac{m-1}{m}-2|}$. Since $C_{k_1,k_2,m-1}$ is nonvanishing only for the case of $k_1+k_2=m$, we have

$$\left\langle w^+(\ell_1)w^\pm(\ell_2)\mathcal{B}_1\right\rangle = \frac{1}{m}\sum_{k}^{m-1}(\pm)^{k-1}\left(\frac{M}{2}\right)\ell_1\ell_2\widetilde{K}_{\frac{k}{m}}(M\ell_1)\widetilde{K}_{1-\frac{k}{m}}(M\ell_2) \ . \tag{4.16}$$

Comparing eqs. (4.16) to (2.2) we obtain the desired relation

$$\left\langle w^+(\ell_1)w^\pm(\ell_2)\mathcal{B}_1\right\rangle = \{\ell_1+(-1)^m\ell_2\}\left\langle w^+(\ell_1)w^\pm(\ell_2)\right\rangle \ . \tag{4.17}$$

Note here that we have $\langle \mathcal{B}_1 w^+(\ell)\rangle = (-1)^m\langle \mathcal{B}_1 w^-(\ell)\rangle$.

Next, let us consider $\mathcal{B}_2$. Since we infer that the insertion of $\mathcal{B}_2$ should play the role of connecting two parts of loops together, we expect the relation

$$\begin{aligned}\left\langle w^+(\ell_1)w^+(\ell_2)\mathcal{B}_2\right\rangle &= 2\ell_1\int_0^{\ell_1} d\ell_1' \ \left\langle w^+(\ell_1')w^+(\ell_2)\right\rangle\left\langle w^+(\ell_1-\ell_1')\right\rangle \\ &+2\ell_2\int_0^{\ell_2} d\ell_2' \ \left\langle w^+(\ell_1)w^+(\ell_2')\right\rangle\left\langle w^+(\ell_2-\ell_2')\right\rangle \\ &+2\ell_1\ell_2\left\langle w^+(\ell_1+\ell_2)\right\rangle \ .\end{aligned} \tag{4.18}$$

The third term in the right hand side of eq. (4.18) represents the contribution from the surfaces with loops $w^+(\ell_1)$ and $w^+(\ell_2)$ touching with each other on a point. Let us confirm the relation (4.18) in the following. In this case, it is convenient to

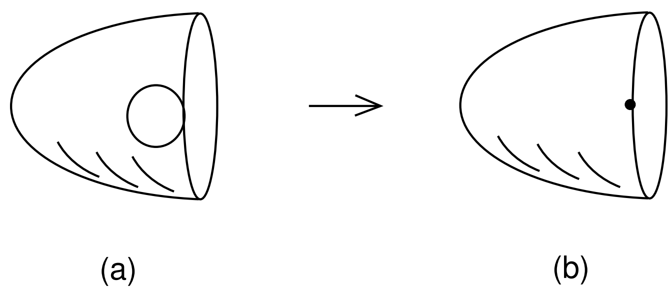

Figure 3: (a): A surface with two loops touching each other on a point. (b): When one of the loops shrinks to a microscopic loop the microscopic loop is equivalent to the insertion of the operator denoted by the dot on the loop.

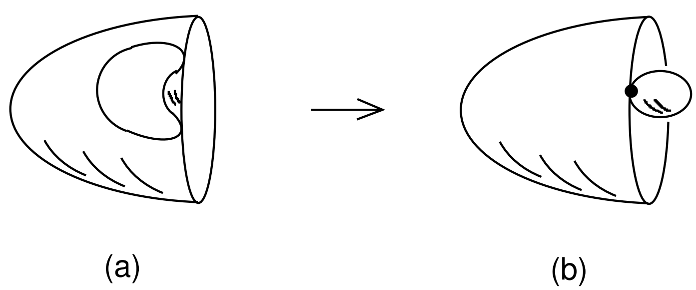

Figure 4: The case of a surface with two loops touching each other on two points.

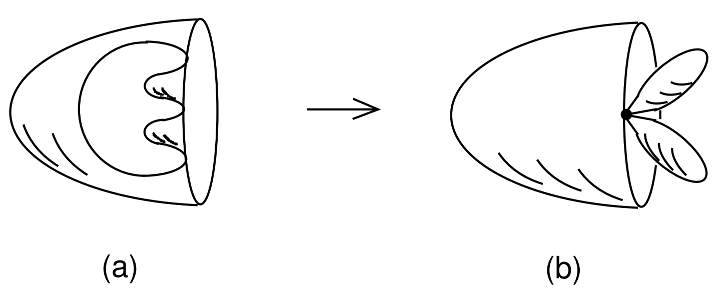

Figure 5: The case of a surface with two loops touching each other on three points.

consider in the space of Laplace transformed coordinates ζ_i. In this space eq. (4.15) reads as

$$\left\langle \hat{W}^+(\zeta_1)\hat{W}^\pm(\zeta_2)\ \widehat{\sigma}_{|k_3+2mn_3|}\right\rangle = \frac{-1}{m+1}\left(\frac{M}{2}\right)^{-\frac{1}{m}-2+|\frac{k_3}{m}+2n_3|}(\frac{k_3}{m}+2n_3)$$
$$\times\frac{\partial}{\partial\zeta_1}\frac{\partial}{\partial\zeta_2}\left\{\sum_{k_1,k_2} C_{k_1k_2k_3}(\pm)^{k_2-1}\ \frac{\sinh(m-k_1)\theta_1}{\sinh m\theta_1}\ \frac{\sinh(m-k_2)\theta_2}{\sinh m\theta_2}\right\}, \tag{4.19}$$

where $\hat{W}^\pm(\zeta_i) \equiv \mathcal{L}[w^\pm(\ell_i)]$. Due to the relation

$$\sum_{k_1,k_2} C_{k_1k_2k_3}(\pm)^{k2}\ \frac{\sinh(m-k_1)\theta_1}{\sinh m\theta_1}\ \frac{\sinh(m-k_2)\theta_2}{\sinh m\theta_2}$$
$$= \frac{-1}{2(\cosh\theta_1 \mp \cosh\theta_2)}\left(\frac{\sinh(m-k_3)\theta_1}{\sinh m\theta_1} - (\pm)^{k_3}\frac{\sinh(m-k_3)\theta_2}{\sinh m\theta_2}\right), \tag{4.20}$$

we have

$$\left\langle \hat{W}^+(\zeta_1)\hat{W}^\pm(\zeta_2)\ \widehat{\sigma}_{|k_3+2mn_3|}\right\rangle = \frac{\pm 1}{2(m+1)}\left(\frac{M}{2}\right)^{-\frac{1}{m}-2+|\frac{k_3}{m}+2n_3|}(\frac{k_3}{m}+2n_3)$$
$$\times\frac{\partial}{\partial\zeta_1}\frac{\partial}{\partial\zeta_2}\left\{\frac{1}{\cosh\theta_1 \mp \cosh\theta_2}\left(\frac{\sinh(m-k_3)\theta_1}{\sinh m\theta_1} - (\pm)^{k_3}\frac{\sinh(m-k_3)\theta_2}{\sinh m\theta_2}\right)\right\}. \tag{4.21}$$

Since we should take $k_3 = 2$ for $\mathcal{B}_2 = -\widehat{\sigma}_{2(m+1)}$ (for $m \geq 3$), we obtain the amplitude for $\mathcal{B}_2$

$$\left\langle \hat{W}^+(\zeta_1)\hat{W}^\pm(\zeta_2)\mathcal{B}_2\right\rangle$$
$$= \frac{\mp 1}{m}\left(\frac{M}{2}\right)^{\frac{1}{m}}\frac{\partial}{\partial\zeta_1}\frac{\partial}{\partial\zeta_2}\left\{\frac{1}{\cosh\theta_1 \mp \cosh\theta_2}\left(\frac{\sinh(m-2)\theta_1}{\sinh m\theta_1} - \frac{\sinh(m-2)\theta_2}{\sinh m\theta_2}\right)\right\}. \tag{4.22}$$

On the other hand, from the amplitudes [4]

$$\left\langle \hat{W}^+(\zeta_1)\hat{W}^+(\zeta_2)\right\rangle = \frac{\partial}{\partial\zeta_1}\frac{\partial}{\partial\zeta_2}\ln\frac{\cosh\theta_1 - \cosh\theta_2}{\cosh m\theta_1 - \cosh m\theta_2}$$
$$= \frac{\partial}{\partial\zeta_2}\left\{\frac{1}{\cosh\theta_1 - \cosh\theta_2}\frac{\sinh\theta_1}{mM\sinh m\theta_1} - \frac{1}{\zeta_1 - \zeta_2}\right\}, \tag{4.23}$$

we obtain the relation

$$-2\frac{\partial}{\partial\zeta_1}\left\{\left\langle \hat{W}^+(\zeta_1)\hat{W}^+(\zeta_2)\right\rangle\left\langle \hat{W}^+(\zeta_1)\right\rangle\right\} + (1 \leftrightarrow 2)$$

$$
\begin{aligned}
&-2\frac{\partial}{\partial\zeta_1}\frac{\partial}{\partial\zeta_2}\left\{\frac{\left\langle\hat{W}^+(\zeta_1)\right\rangle-\left\langle\hat{W}^+(\zeta_2)\right\rangle}{\zeta_1-\zeta_2}\right\} \\
&=\frac{2}{m}\left(\frac{M}{2}\right)^{\frac{1}{m}}\frac{\partial}{\partial\zeta_1}\frac{\partial}{\partial\zeta_2}\left\{\frac{1}{\cosh\theta_1-\cosh\theta_2}\left(\frac{\sinh\theta_1\cosh(m+1)\theta_1}{\sinh m\theta_1}-(1\leftrightarrow 2)\right)\right\}.
\end{aligned}
\tag{4.24}
$$

One can easily show that the right hand side of eq. (4.24) agrees with that of eq. (4.22). Putting eqs. (4.24) and (4.22) together and performing the inverse Laplace transformation, we obtain the desired relation eq. (4.18).

As for $\langle w^+(\ell_1)w^-(\ell_2)\mathcal{B}_2\rangle$, from the amplitude [4]

$$
\left\langle\hat{W}^+(\zeta_1)\hat{W}^-(\zeta_2)\right\rangle=\frac{\partial}{\partial\zeta_1}\frac{\partial}{\partial\zeta_2}\ln(\cosh\theta_1+\cosh\theta_2)\ , \tag{4.25}
$$

we obtain the relation

$$
\begin{aligned}
\left\langle w^+(\ell_1)w^-(\ell_2)\mathcal{B}_2\right\rangle &= -2\ell_1\int_0^{\ell_1}d\ell_1'\ \left\langle w^+(\ell_1')w^-(\ell_2)\right\rangle\left\langle w^+(\ell_1-\ell_1')\right\rangle \\
&\quad +2\ell_2\int_0^{\ell_2}d\ell_2'\ \left\langle w^+(\ell_1)w^-(\ell_2')\right\rangle\left\langle w^-(\ell_2-\ell_2')\right\rangle
\end{aligned}
\tag{4.26}
$$

in a similar way. In this case, the operator $\mathcal{B}_2$ does not connect the different kinds of loops $w^+(\ell_1)$ and $w^-(\ell_2)$ together.

We have shown that the operator $\mathcal{B}_2$ connects two parts of the same kind of loops together in the case with two loops. We infer that similar phenomena occur in general; the operator $\mathcal{B}_n$ connects n parts of the same kind of loops together in the case with any number of loops.

4.2 Connection to the Schwinger-Dyson equations

We can observe a close relationship between the boundary operators and the Schwinger-Dyson equations proposed in [17]. Continuum limit of the Schwinger-Dyson equations for loops in the two- and multi-matrix models were proposed in [17] under some assumptions. It was shown [17] that these equations for the two-matrix model contain W_3 constraints, which were derived explicitly in [18]. The integrability of these equations were shown in [19]. These facts justify the proposed Schwinger-Dyson equations.

Let us consider the connection of the boundary operators with the Schwinger-Dyson equations. For the $(m+1,m)$ minimal models, the following Schwinger-Dyson equations were proposed in [17]:

$$
\int_0^{\ell}d\ell'\left\langle w^{(1)}(\ell')w^{(1)}\left(\ell-\ell';[\mathcal{H}(\sigma)]^j\right)w^{(1)}(\ell_1)\cdots w^{(1)}(\ell_n)\right\rangle'
$$

$$+g\sum_i \ell_i \left\langle w^{(1)}\left(\ell+\ell_i;[\mathcal{H}(\sigma)]^j\right) w^{(1)}(\ell_1)\cdots w^{(1)}(\ell_{i-1})w^{(1)}(\ell_{i+1})\cdots w^{(1)}(\ell_n)\right\rangle'$$
$$+\left\langle w^{(1)}\left(\ell;[\mathcal{H}(\sigma)]^{j+1}\right) w^{(1)}(\ell_1)\cdots w^{(1)}(\ell_n)\right\rangle' \approx 0 \ ,$$
$$\text{for } \ j=0,\cdots,m-2 \ , \tag{4.27}$$

and

$$\left\langle w^{(1)}\left(\ell;[\mathcal{H}(\sigma)]^{m-1}\right) w^{(1)}(\ell_1)\cdots w^{(1)}(\ell_n)\right\rangle' \approx 0 \ . \tag{4.28}$$

Here $\langle\cdots\rangle'$ represent loop correlators that are not necessarily connected, and $w^{(1)}(\ell)$ represents a loop operator corresponding to a loop created by the matrix $\hat{A}^{(1)}$ in the multi-matrix model. The operator $\mathcal{H}(\sigma)$ describes an operator which changes the 'spin' on a loop locally from 1 to 2. Also $w^{(1)}\left(\ell;[\mathcal{H}(\sigma)]^j\right)$ describes a loop with $[\mathcal{H}(\sigma)]^j$ inserted. The symbol $\approx$ means that as a function of ℓ, the quantity has its support at $\ell=0$.

From eq. (4.27) for $j=0$ and $n=1$, we have the relation

$$\ell_1 \left\langle w^{(1)}\left(\ell_1;\mathcal{H}(\sigma)\right) w^{(1)}(\ell_2)\right\rangle' + \ell_2 \left\langle w^{(1)}(\ell_1) w^{(1)}\left(\ell_2;\mathcal{H}(\sigma)\right)\right\rangle'$$
$$+\ell_1\int_0^{\ell_1} d\ell_1' \left\langle w^{(1)}(\ell_1')w^{(1)}(\ell_1-\ell_1')w^{(1)}(\ell_2)\right\rangle'$$
$$+\ell_2\int_0^{\ell_2} d\ell_2' \left\langle w^{(1)}(\ell_1)w^{(1)}(\ell_2')w^{(1)}(\ell_2-\ell_2')\right\rangle'$$
$$+2g\ell_1\ell_2\left\langle w^{(1)}(\ell_1+\ell_2)\right\rangle' \approx 0 \ . \tag{4.29}$$

The planar part of the above relation agrees with eq. (4.18). Note that the loop amplitudes in eq. (4.18) represent connected correlators.

This agreement implies that $\mathcal{H}$ would correspond to $\widehat{\mathcal{B}}_2$. Taking into account the fact that $\widehat{\mathcal{B}}_n$ ($n=0$ mod m) do not exist and eq. (4.28), it is legitimate to consider that the amplitude (for $j=1,\cdots,m$)

$$\left\langle w^+(\ell_1)\cdots w^+(\ell_n)\widehat{\mathcal{B}}_j\right\rangle \tag{4.30}$$

corresponds to the connected part of the amplitude

$$\sum_{i=1}^n \oint d\sigma_i \left\langle w^{(1)}(\ell_1)\cdots w^{(1)}(\ell_{i-1})w^{(1)}\left(\ell_i;[\mathcal{H}(\sigma_i)]^{j-1}\right) w^{(1)}(\ell_{i+1})\cdots w^{(1)}(\ell_n)\right\rangle' \ . \tag{4.31}$$

5 Summary

In this paper we have examined the correlators in the $(m+1,m)$ unitary minimal models coupled to two-dimensional gravity from the point of view of the two-matrix

model. From the two-loop correlators and the wave function of the scaling operators, we derived the explicit form of the expansion of the loops in terms of the scaling operators. Using this expansion, we deduced the three-point functions from the three loop operators, and showed that simple fusion rules exist for all of the scaling operators. The three-loop correlator [13] can be understood to express these fusion rules in a compact form.

At the $(m+1,m)$ critical point in two-matrix models, the scaling operators $\widehat{\sigma}_j$ ($j=0 \bmod m+1$) have no counterparts in the BRST cohomology of Liouville theory coupled to the corresponding conformal matter. In [15],these operators were argued to be boundary operators which couple to loops in the case of the one-matrix model. It was also shown explicitly that one of them, corresponding to $\widehat{\sigma}_{m+1}$ in the case of the unitary matter, is a operator which measures the total length of the loops.

We examined the role played by the rest of these operators. We showed, in some examples, that the operator $\mathcal{B}_n$ couples to the points to which n parts of the same kind of loops are stuck each other. In other words, the operator $\mathcal{B}_n$ connects n parts of the same kind of loops together. We think these operators play an important role concerning the touching of the macroscopic loops. The emergence of these operators in matrix models can easily be understood from the viewpoint of macroscopic loops and their expansion in terms of local operators.

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Appendix A

Let us prove that the two-loop correlator eq. (2.2) can be written as eq. (2.7). From eq. (2.2), we have

$$\left\langle w^{+}(\ell_1)w^{\pm}(\ell_2)\right\rangle = \sum_{k=1}^{m-1}(\pm)^{k-1}\left(\frac{\sin\pi\frac{k}{m}}{\pi/2}\right)^2 \left\langle w_k(\ell_1)w_k(\ell_2)\right\rangle \tag{A.1}$$

and

$$\frac{\partial}{\partial M}\left\langle w_k(\ell_1)w_k(\ell_2)\right\rangle = -\frac{1}{m}\frac{M}{2}\ell_1\ell_2\ K_{1-\frac{k}{m}}(M\ell_1)\ K_{1-\frac{k}{m}}(M\ell_2)\ , \tag{A.2}$$

where we have introduced loop operators $w_k(l)$ which represent loops with some distinct matter boundary condition. Making use of a formula

$$K_\nu(z)\ K_\nu(w) = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}t}{t}\ K_\nu\left(\frac{zw}{t}\right)\ \exp\left(-\frac{t}{2}-\frac{z^2+w^2}{2t}\right) \tag{A.3}$$

and replacing t with tM^2, we have

$$\begin{aligned} &M\ell_1\ell_2\ K_{1-\frac{k}{m}}(M\ell_1)\ K_{1-\frac{k}{m}}(M\ell_2) \\ &\quad = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}t}{t}\ K_{1-\frac{k}{m}}\left(\frac{\ell_1\ell_2}{t}\right)\ \exp\left(-\frac{tM^2}{2}-\frac{\ell_1^2+\ell_2^2}{2t}\right)\quad . \end{aligned} \tag{A.4}$$

Carrying out the integral with respect to M, and from eq. (A.2), we have

$$\begin{aligned} &\langle w_k(\ell_1)w_k(\ell_2)\rangle \\ &\quad = \frac{1}{4m}\int_0^\infty \frac{\mathrm{d}t}{t}\ \frac{\ell_1\ell_2}{t}K_{1-\frac{k}{m}}\left(\frac{\ell_1\ell_2}{t}\right)\ \exp\left(-\frac{tM^2}{2}-\frac{\ell_1^2+\ell_2^2}{2t}\right)\ . \end{aligned} \tag{A.5}$$

Due to a formula,

$$zK_{1-|p|}(z) = \int_0^\infty \mathrm{d}E\frac{E\sinh\pi E}{\cosh\pi E-\cos\pi p}\ K_{iE}(z)\ \ , \tag{A.6}$$

the right hand side of eq. (A.5) turns into

$$\begin{aligned} &\frac{1}{4m}\int_0^\infty \frac{\mathrm{d}t}{t} \\ &\quad\times\int_0^\infty \mathrm{d}E\frac{E\sinh\pi E}{\cosh\pi E-\cos\pi p}\ K_{iE}\left(\frac{\ell_1\ell_2}{t}\right)\exp\left(-\frac{tM^2}{2}-\frac{\ell_1^2+\ell_2^2}{2t}\right). \end{aligned} \tag{A.7}$$

Using a formula eq. (A.3) again, eq. (A.7) turns out to be

$$\frac{1}{2m}\int_0^\infty \mathrm{d}E\frac{E\sinh\pi E}{\cosh\pi E-\cos\pi\frac{k}{m}}\ K_{iE}(M\ell_1)\ K_{iE}(M\ell_2)\ . \tag{A.8}$$

Putting eq. (A.8) and eq. (A.1) together, we have

$$\begin{aligned} \left\langle w^+(\ell_1)w^\pm(\ell_2)\right\rangle &= \sum_{k=1}^{m-1}\frac{1}{2m}(\pm)^{k-1}\left(\frac{\sin\pi\frac{k}{m}}{\pi/2}\right)^2 \\ &\quad\times\int_0^\infty \mathrm{d}E\frac{E\sinh\pi E}{\cosh\pi E-\cos\pi\frac{k}{m}}\ K_{iE}(M\ell_1)\ K_{iE}(M\ell_2) \end{aligned} \tag{A.9}$$

The integral in E can be carried out by deforming the contour. The residues for poles $E=\pm i(\frac{k}{m}+2n),\ n=0,\pm1,\pm2,\cdots$, contribute to the integral and, after all, we obtain the following expansion for the two-loop correlators (for $\ell_1<\ell_2$)

$$\begin{aligned} &\left\langle w^+(\ell_1)w^\pm(\ell_2)\right\rangle \\ &\quad = \frac{1}{m}\sum_{k=1}^{m-1}\sum_{n=-\infty}^{\infty}(\pm)^{k-1}\left|\frac{k}{m}+2n\right|\ I_{|\frac{k}{m}+2n|}(M\ell_1)\ \widetilde{K}_{\frac{k}{m}+2n}(M\ell_2)\ . \end{aligned} \tag{A.10}$$

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